ON SEMI-INFINITE COHOMOLOGY OF FINITE DIMENSIONAL ALGEBRAS

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Abstract. We show that semi-infinite cohomology of a finite dimensional graded algebra (satisfying some additional requirements) is a particular case of a general categorical construction. An example of this situation is provided by small quantum groups at a root of unity.

1. Introduction

Semi-infinite cohomology of associative algebras was studied, in particular, by S. Arkhipov (see [Ar1], [Ar2], [Ar3]). Recall that the definition of semi-infinite cohomology in [Ar1] works in the following set-up. We are given an associative graded algebra A, two subalgebras $B, N \subset A$ such that $A = B \otimes N$ as a vector space, satisfying some additional assumptions. In this situation the space of semi-infinite Ext's, $Ext^{\infty/2+\bullet}(X,Y)$ is defined for X, Y in the bounded derived category of graded A-modules. The definition makes use of explicit complexes.

In this note we show that under some additional assumptions semi-infinite Ext groups $Ext^{\infty/2+\bullet}(X,Y)$ has a categorical interpretation. More precisely, given a category $\mathcal{A}$ and subcategory $\mathcal{B} \subset \mathcal{A}$ one can define for $X, Y \in \mathcal{A}$ the set of morphisms from X to Y "through $\mathcal{B}$"; we denote this space by $Hom_{\mathcal{A}_\mathcal{B}}(X,Y)$. We then show that if $\mathcal{A}$ is the bounded derived category of A-modules, and $\mathcal{B}$ is the full triangulated subcategory generated by B-projective A-modules, then, under certain assumptions one has

$$Ext^{\infty/2+i}(X,Y) = Hom_{\mathcal{A}_\mathcal{B}}(X, Y[i]). \tag{1}$$

Notice that the right hand side of (1) makes sense for a wide class of pairs (A, B) (an associative algebra, and a subalgebra), and $X, Y \in D^b(A - mod)$; in particular we do not need A, B to be graded. Thus one may consider (1) as providing a generalization of the definition of semi-infinite Ext's to this set up. However, we should warn the reader that under our working assumptions, but not in general, $\mathcal{B}$ also equals the full triangulated subcategory generated by B-injective modules, or by modules (co)induced from a "complemental" subalgebra $N \subset A$, so one has at least four different obvious generalizations of the definition of the right-hand side of (1).

In fact, a description of semi-infinite cohomology similar to (1) in a general situation (in particular, in the case of enveloping algebras of infinite-dimensional Lie algebras) requires additional ideas, and is the subject of a forthcoming joint work with Arkhipov and Positselskii.

An example of the situation considered in this paper is provided by a small quantum group at a root of unity [L], or by the restricted enveloping algebra of a simple Lie algebra in positive characteristic. Computation of semi-infinite cohomology in

the former case is due to S. Arkhipov [Ar1] (the answer suggested as a conjecture by B. Feigin). This example was a motivation for the present work. We informally explain the relation of our Theorem 1 to the answer for semi-infinite cohomology of small quantum groups in Remark 5 below (we plan to derive it from Theorem 1 elsewhere).

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2. Categorical preliminaries: morphisms through a functor

Let $\mathcal{A}$, $\mathcal{B}$ be (small) categories, and $\Phi : \mathcal{B} \to \mathcal{A}$ be a functor. For $X, Y \in Ob(\mathcal{A})$ define the set of "morphisms from X to Y through Φ" as π_0 of the category of diagrams

$$X \longrightarrow \Phi(?) \longrightarrow Y, \qquad ? \in \mathcal{B}. \tag{2}$$

This set will be denoted by $Hom_{\mathcal{A}_\Phi}(X,Y)$. Thus elements of $Hom_{\mathcal{A}_\Phi}(X,Y)$ are diagrams of the form (2), with two diagrams identified if there exists a morphism between them. Composing the two arrows in (2) we get a functorial map

$$Hom_{\mathcal{A}_\Phi}(X,Y) \longrightarrow Hom_{\mathcal{A}}(X,Y). \tag{3}$$

If $\mathcal{A}$, $\mathcal{B}$ are additive and Φ is an additive functor, then addition of diagrams of the form (2) is defined by

$$(X \xrightarrow{f} \Phi(Z) \xrightarrow{g} Y) + (X \xrightarrow{f'} \Phi(Z') \xrightarrow{g'} Y) = (X \xrightarrow{f\times f'} \Phi(Z \oplus Z') \xrightarrow{g\oplus g'} Y);$$

it induces an abelian group structure on $Hom_{\mathcal{A}_\Phi}(X,Y)$. Proposition 3 in [ML], VIII.2 shows that for $Z \in \mathcal{B}$ the tautological map

$$Hom(X, \Phi(Z)) \otimes_{\mathbb{Z}} Hom(\Phi(Z), Y) \to Hom_{\mathcal{A}_\Phi}(X,Y)$$

is compatible with addition.

We have the composition map

$$Hom_{\mathcal{A}}(X', X) \times Hom_{\mathcal{A}_\Phi}(X,Y) \times Hom_{\mathcal{A}}(Y,Y') \to Hom_{\mathcal{A}_\Phi}(X',Y);$$

in particular, for $\mathcal{A}, \mathcal{B}, \Phi$ additive, $Hom_{\mathcal{A}_\Phi}(X,Y)$ is an $End(X) - End(Y)$ bimodule.

Given $\Phi : \mathcal{A} \to \mathcal{B}$, $\Phi' : \mathcal{A}' \to \mathcal{B}'$ and $F : \mathcal{A} \to \mathcal{A}'$, $G : \mathcal{B} \to \mathcal{B}'$ with $F \circ \Phi \cong \Phi' \circ G$ we get for $X, Y \in \mathcal{A}$ a map

$$Hom_{\mathcal{A}_\Phi}(X,Y) \to Hom_{\mathcal{A}'_{\mathcal{B}'}}(F(X), F(Y)). \tag{4}$$

If the left adjoint functor Φ^* to Φ is defined on X, then we have

$$Hom_{\mathcal{A}_\Phi}(X,Y) = Hom_{\mathcal{A}}(\Phi(\Phi^*(X)), Y),$$

because in this case the above category contracts to the subcategory of diagrams of the form $X \xrightarrow{can} \Phi(\Phi^*(X)) \to Y$, where *can* stands for the adjunction morphism. If the right adjoint functor $\Phi^!$ is defined on Y, then

$$Hom_{\mathcal{A}_\Phi}(X,Y) = Hom_{\mathcal{A}}(X, \Phi(\Phi^!(Y)))$$

for similar reasons. In particular, if Φ is a full imbedding then (3) is an isomorphism provided either X or Y lie in the image of Φ.

In all examples below $\mathcal{A}$ will be a triangulated category, and $\Phi : \mathcal{B} \to \mathcal{A}$ will be an imbedding of a (strictly) full triangulated subcategory. Given $\mathcal{B} \subset \mathcal{A}$ we will

tacitly assume Φ to be the imbedding, and write $Hom_{\mathcal{A}_\mathcal{B}}$ ("morphisms through $\mathcal{B}$") instead of $Hom_{\mathcal{A}_\Phi}$.

Example 1. Let M be a Noetherian scheme, and $\mathcal{A} = D^b(Coh_M)$ be the bounded derived category of coherent sheaves on M; let $I : \mathcal{B} \hookrightarrow \mathcal{A}$ be the full subcategory of complexes whose cohomology ia supported on a closed subset $i : N \hookrightarrow M$. Then the functor $I \circ I^! = i_* \circ i^!$ takes values in a larger derived category of quasi-coherent sheaves (i.e. ind-coherent sheaves), and $I \circ I^* = i_* \circ i^*$ takes values in the Grothendick-Serre dual category, the derived category of pro-coherent sheaves (introduced in Deligne's appendix to [H]). Still we have

$$Hom_{\mathcal{A}_\mathcal{B}}(X, Y) = Hom(X, i_*(i^!(Y))) = Hom(i_*(i^*(X)), Y).$$

In particular, if $X = \mathcal{O}_M$ is the structure sheaf, we get

$$Hom_{\mathcal{A}_\mathcal{B}}(\mathcal{O}_M, Y[i]) = H^i_N(Y), \tag{5}$$

where $H^\bullet_N(Y)$ stands for cohomology with support on N (local cohomology) [H].

3. Recollection of the definition of $Ext^{\infty/2+\bullet}$

All algebras below will be associative and unital algebras over a field.

We recall a variant of definition of semi-infinite Ext's (available under certain restrictions on the algebra and subalgebras) suited for our purpose (see e.g. [FS], §2.4, pp 180-183, for this definition in the particular case of small quantum groups; the general case is analogous).

We make the following assumptions. A $\mathbb{Z}$-graded algebra A and graded subalgebras A^0, $A^{\le 0}$, $A^{\ge 0} \subset A$ are fixed and satisfy the following conditions:

(1) $A^{\le 0}$, $A^{\ge 0}$ are graded by, respectively, $\mathbb{Z}^{\le 0}$, $\mathbb{Z}^{\ge 0}$, and $A^0 = A^{\le 0} \cap A^{\ge 0}$ is the component of degree 0 in $A^{\ge 0}$ and in $A^{\le 0}$.

(2) The maps $A^{\ge 0} \otimes_{A^0} A^{\le 0} \to A$ and $A^{\le 0} \otimes_{A^0} A^{\ge 0} \to A$ provided by the multiplication map are isomorphisms.

(3) A is finite dimensional; A^0 is semisimple, and $A^{\ge 0}$ is self-injective (i.e. the free $A^{\ge 0}$-module is injective).

By a "module" we will mean a finite dimensional graded module, unless stated otherwise. By $A - mod$ we denote the category of (graded finite dimensional) A-modules.

Recall that a bounded below complex of graded modules is called *convex* if the weights "go down", i.e. for any $n \in \mathbb{Z}$ the sum of weight spaces of degree more than n is finite dimensional. A bounded below complex of graded modules is called *concave* if the weights "go up" in the similar sense.

Lemma 1. *i) Any A-module admits a right convex resolution by A-modules, which are injective as $A^{\ge 0}$-modules. It also admits a right concave resolution by A-modules, which are $A^{\le 0}$-injective.*

ii) Any finite complex of A-modules is a quasiisomorphic subcomplex of a bounded below convex complex of $A^{\ge 0}$-injective A-modules. It is also a quasiisomorphic subcomplex of a bounded below concave complex of $A^{\le 0}$-injective A-modules.

Proof. To deduce (ii) from (i) imbed given finite complex $C^\bullet \in Com(A - mod)$ into a complex of A-injective modules $I^\bullet \in Com^{\ge 0}(A - mod)$ (notice that condition (2) above implies that an A-injective module is also $A^{\ge 0}$ and $A^{\le 0}$ injective), and apply (i) to the module of cocycles $Z^n = I^n / d(I^{n-1})$ for large n.

To check (i) it suffices to find for any $M \in A - mod$ an imbedding $M \hookrightarrow I$, where I is $A^{\leq 0}$ injective, and if n is such that all graded components M_i for $i < n$ vanish, then $M_n \widetilde{\longrightarrow} I_n$. (This would prove the second part of the statement; the first one is obtained from the first one by renotation.) It suffices to take $I = CoInd_{A^{\geq 0}}^{A}(Res_{A^{\geq 0}}^{A}(M))$. It is indeed $A^{\leq 0}$-injective, because of the equality

$$Res_{A^{\leq 0}}^{A}(CoInd_{A^{\geq 0}}^{A}(M)) = CoInd_{A^{0}}^{A^{\leq 0}}(M)), \tag{6}$$

which is a consequence of assumption (2) above. □

We set $D = D^b(A - mod)$.

Definition 1. (cf. [FS], §2.4) The assumptions (1–3) are enforced. Let $X, Y \in D$. Let $J^X_\searrow$ be a convex bounded below complex of $A^{\geq 0}$-injective (= projective) modules quasiisomorphic to X, and $J^Y_\nearrow$ be a concave bounded below complex of $A^{\leq 0}$-injective modules quasiisomorphic to Y. Then one defines

$$Ext^{\infty/2+i}(X, Y) = H^i(Hom^\bullet(J^X_\searrow, J^Y_\nearrow)). \tag{7}$$

Remark 1. Independence of the right-hand side of (7) on the choice of resolutions $J^X_\searrow$, $J^Y_\nearrow$ follows from the argument below. Since particular complexes used in [Ar1] to define $Ext^{\infty/2+\bullet}$ satisfy our assumptions, we see that this definition agrees with the one in *loc. cit.*

Remark 2. Notice that Hom in the right-hand side of (7) is Hom in the category of graded modules. As usual, it is often convenient to denote by $Ext^{\infty/2+i}(X, Y)$ the graded space which in present notations is written down as $\bigoplus\limits_n Ext^{\infty/2+i}(X, Y(n))$, where (n) refers to shift of grading by $-n$.

Remark 3. The next standard Lemma shows that conditions on the resolutions $J^X_\searrow$, $J^Y_\nearrow$ used in the (7) can be formulated in terms of the subalgebra $A^{\geq 0}$ alone (or, alternatively, in terms of $A^{\leq 0}$ alone); this conforms with the fact that the left-hand side of (11) in Theorem 1 below depends only on $A^{\geq 0}$. However, existence of a "complemental" subalgebra $A^{\leq 0}$ is used in the construction of a resolution $J^X_\searrow$ with required properties.

Lemma 2. *An A-module is $A^{\leq 0}$-injective iff it is has a filtration with subquotients of the form $CoInd_{A^{\geq 0}}^{A}(M)$, $M \in A^{\geq 0} - mod$.*

Proof. The "if" direction follows from semisimplicity of A^0, and equality (6) above. To show the "only if" part let M be an $A^{\leq 0}$-injective A-module. Let M^- be its graded component of minimal degree; then the canonical morphism

$$M \to CoInd_{A^0}^{A^{\leq 0}}(M^-) \tag{8}$$

is surjective. If M is actually an A-module, then the projection $M \to M^-$ is a surjection of $A^{\geq 0}$-modules, hence yields a morphism

$$M \to CoInd_{A^{\geq 0}}^{A}(M^-). \tag{9}$$

(6) shows that $Res_{A^{\leq 0}}^{A}$ sends (9) into (8); in particular (9) is surjective. Thus the top quotient of the required filtration is constructed, and the proof is finished by induction. □

Remark 4. In two special cases $Ext^{\infty/2+i}(X,Y)$ coincides with a traditional derived functor. First, suppose that $Res^A_{A^{\geq 0}}(X)$ has finite injective (equivalently, projective) dimension; then one can use a finite complex $J^X_{\searrow}$ in (7) above. It follows immediately, that in this case we have

$$Ext^{\infty/2+i}(X,Y) \cong Hom(X, Y[i]).$$

On the other hand, suppose that $Res^A_{A^{\leq 0}}(Y)$ has finite injective dimension, so that the complex $J^Y_{\nearrow}$ in (7) can be chosen to be finite. To describe semi-infinite Ext's in this case we need another notation. Let A^* denote the co-regular A-bimodule; for $M \in A-mod$ let $M\check{} = M^* = Hom_A(M, A^*)$ denote the corresponding right A-module, and we use the same notation for the corresponding functor on the derived categories. Let also $S : D^b(A-mod) \to D^+(A-mod)$ be given by $S(Y) = RHom_A(A^*, Y)$. Notice that A^* is $A^{\geq 0}$-projective by self-injectivity of $A^{\geq 0}$; thus Lemma 2 shows that $Ext^i_A(A^*, N) = 0$ for $i > 0$ if N is $A^{\leq 0}$-injective. In particular, $S(Y) \in D^b(A-mod)$ if $Y|_{A^{\leq 0}}$ has finite injective dimension. We claim that in this case we have

$$Ext^{\infty/2+i}(X,Y) \cong X\check{} \overset{L}{\otimes}_A S(Y).$$

This isomorphism an immediate consequence of the next Lemma. We also remark that if A is a Frobenius algebra, then $S \cong Id$.

Lemma 3. *Let $M, N \in A-mod$ be such that M is $A^{\geq 0}$-projective, while N is $A^{\leq 0}$-injective. Then we have*

a) $Ext^i_A(M,N) = 0$; $Ext^i_A(A^*, N) = (R^iS)(N) = 0$, $Tor^A_i(M\check{}, S(N)) = 0$ for $i \neq 0$.

b) The natural map

$$M\check{} \otimes_A S(N) = Hom_A(M, A^*) \otimes_A Hom_A(A^*, N) \longrightarrow Hom_A(M,N) \tag{10}$$

is an isomorphism.

Proof. The first equality in (a) follows from Lemma 2, and the second one was checked above. Self-injectivity of $A^{\geq 0}$ shows that $M\check{}$ is $A^{\geq 0}$-projective, and a variant of Lemma 2 ensures that it is filtered by modules induced from $A^{\leq 0}$. Thus it sufficies to show that $S(N)$ is $A^{\leq 0}$-projective. This follows from isomorphisms

$$Hom_A(A^*, CoInd^A_{A^{\geq 0}}(N_0)) = Hom_{A^{\geq 0}}(A^*, N_0) \cong Hom_{A^{\geq 0}}((A^{\geq 0})^*, N_0) \otimes_{A^0} A^{\leq 0}.$$

Let us now deduce (b) from (a). Notice that (a) implies that both sides of (10) are exact in N (and also in M), i.e. send exact sequences $0 \to N' \to N \to N'' \to 0$ with N', N'' being $A^{\leq 0}$-injective into exact sequences. Also (10) is evidently an isomorphism for $N = A^*$. For any $A^{\leq 0}$-injective N there exists an exact sequence

$$0 \to N \to (A^*)^n \xrightarrow{\phi} (A^*)^m$$

with image and cokernel of ϕ being $A^{\leq 0}$-injective. Thus both sides of (10) turn it into an exact sequence, which shows that (10) is an isomorphism for any $A^{\leq 0}$-injective N. $\square$

4. Main result

Theorem 1. *Let $D_{\infty/2} \subset D$ be the full tringulated subcategory of D generated by $A^{\geq 0}$-injective (=projective) modules. For $X, Y \in D^b(A-mod)$ we have a natural isomorphism*

$$Hom_{D_{D_{\infty/2}}}(X, Y[i]) \cong Ext^{\infty/2+i}(X, Y). \tag{11}$$

The proof of Theorem 1 is based on the following

Lemma 4. *i) Every graded $A^{\geq 0}$-injective A-module admits a concave right resolution consisting of A-injective modules.*

ii) A finite complex of graded $A^{\geq 0}$-injective A-modules is quasiisomorphic to a concave bounded below complex of A-injective modules.

Proof. (ii) follows from (i) as in the proof of Lemma 1. (Recall that, according to Hilbert, if a bounded below complex of injectives represents an object $X \in D^b$ which has finite injective dimension, then for large n the module of cocycles is injective.)

To prove (i) it is enough for any $A^{\geq 0}$-injective module M to find an imbedding $M \hookrightarrow I$, where I is A-injective, and $M_n \widetilde{\longrightarrow} I_n$ provided $M_i = 0$ for $i < n$. (Notice that cokernel of such an imbedding is $A^{\geq 0}$-injective, because I is $A^{\geq 0}$-injective by condition (2).) We can take I to be $CoInd^A_{A_{\geq 0}}(Res^A_{A_{\geq 0}}(M))$. Then I is indeed injective, because M is $A^{\geq 0}$-injective by semi-simplicity of A^0, and condition on weights is clearly satisfied. □

Proposition 1. *a) Let $J_\searrow$ be a convex bounded below complex of A-modules. Let $J^n_\searrow$ be the n-th stupid truncation of $J_\searrow$ (thus $J^n_\searrow$ is a quotient complex of $J_\searrow$).*

Let Z be a finite complex of $A^{\geq 0}$-injective A-modules. Then we have

$$Hom_D(X, Z) \widetilde{\longrightarrow} \varinjlim Hom_D(J^n_\searrow, Z). \tag{12}$$

In fact, for n large enough we have

$$Hom_D(X, Z) \widetilde{\longrightarrow} Hom_D(J^n_\searrow, Z).$$

Proof. Let $I_\nearrow$ be a concave bounded below complex of A-injective modules quasi-isomorphic to Z (which exists by Lemma 4(ii)). Then the left-hand side of (12) equals $Hom_{Hot}(J_\searrow, I_\nearrow)$ where Hot stands for the homotopy category of complexes of A-modules. Conditions on weights of our complexes ensure that there are only finitely many degrees for which the corresponding graded components both in $J_\searrow$ and $I_\nearrow$ are nonzero; thus any morphism between graded vector spaces $J_\searrow$, $I_\nearrow$ factors through the finite dimensional sum of corresponding graded components. In particular, $Hom^\bullet(J^n_\searrow, I_\nearrow) \widetilde{\longrightarrow} Hom^\bullet(J_\searrow, I_\nearrow)$ for large n, and hence

$$Hom_{D(A-mod)}(J^n_\searrow, I_\nearrow) = Hom_{Hot}(J^n_\searrow, I_\nearrow) \widetilde{\longrightarrow} Hom_{Hot}(J_\searrow, I_\nearrow)$$

for large n. □

Proof of the Theorem. We keep notations of Definition 1. It follows from the Proposition that

$$Hom_{D_{D_{\infty/2}}}(X, Y[i]) = \varinjlim{}_n Hom_D((J^X_\searrow)^n, Y[i]).$$

The right-hand side of (11) (defined in (7)) equals $H^i(Hom^\bullet(J^X_\searrow, J^Y_\nearrow))$. Conditions on weights of $J^X_\searrow$, $J^Y_\nearrow$ show that for large n we have

$$Hom^\bullet((J^X_\searrow)^n, J^Y_\nearrow) \widetilde{\longrightarrow} Hom^\bullet(J^X_\searrow, J^Y_\nearrow).$$

Lemma 2 implies that $Ext^i_A(M_1, M_2) = 0$ for $i > 0$ if M_1 is $A^{\geq 0}$-projective, and M_2 is $A^{\leq 0}$-injective. Thus

$$Hom_D((J^X_{\searrow})^n, Y[i]) = H^i(Hom^\bullet(J^X_{\searrow}, J^Y_{\nearrow})).$$

The Theorem is proved. □

Remark 5. This remark concerns with the example provided by a small quantum group. So let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $q \in \mathbb{C}$ be a root of unity of order l, and let $A = u_q = u_q(\mathfrak{g})$ be the corresponding small quantum group [L]. Let $A^{\geq 0} = b_q \subset u_q$ and $A^{\leq 0} = b_q^- \subset u_q$ be respectively the upper and the lower triangular subalgebras. Then the above conditions (1–3) are satisfied.

Let $\mathbb{I}$ denote the trivial u_q-module. The cohomology $Ext^\bullet_{u_q}(\mathbb{I}, \mathbb{I})$, and the semi-infinite cohomology $Ext^{\infty/2+\bullet}(\mathbb{I}, \mathbb{I})$ were computed respectively in [GK] and [Ar1]. Let us recall the results of these computations.

Assume for simplicity that l is prime to twice the maximal multiplicity of an edge in the Dynkin diagram of $\mathfrak{g}$. Let $\mathcal{N} \subset \mathfrak{g}$ be the cone of nilpotent elements, and $\mathfrak{n} \subset \mathcal{N}$ be a maximal nilpotent subalgebra. Then the Theorem of Ginzburg and Kumar asserts that

$$Ext^\bullet(\mathbb{I}, \mathbb{I}) \cong \mathcal{O}(\mathcal{N}), \tag{13}$$

the algebra of regular functions on $\mathcal{N}$. Also, a Theorem of Arkhipov (conjectured by Feigin) asserts that

$$Ext^{\infty/2+\bullet}(\mathbb{I}, \mathbb{I}) \cong H^d_{\mathfrak{n}}(\mathcal{N}, \mathcal{O}), \tag{14}$$

where d is the dimension of $\mathfrak{n}$, and $H_{\mathfrak{n}}$ denotes cohomology with support on $\mathfrak{n}$; one also has $H^i_{\mathfrak{n}}(\mathcal{N}, \mathcal{O}) = 0$ for $i \neq d$ (here the choice of $\mathfrak{n}$ is assumed to be compatible with the choice of an upper triangular subalgebra $b_q \subset u_q$ via isomorphism (13) in a natural sense).

The aim of this remark is to point out a formal similarity between (14) and equality (5) in Example 1 above. Namely, the Ginzburg-Kumar isomorphism (13) yields a functor $F : D^b(u_q - mod) \to Coh(\mathcal{N})$, $F(X) = Ext^\bullet(\mathbb{I}, X)$, such that $F(\mathbb{I}) = \mathcal{O}_{\mathcal{N}}$ is the structure sheaf. It is easy to see that if $X \in D^b(u_q - mod)$ has finite projective (equivalently, injective) homological dimension over b_q, then the support of $F(X)$ lies in $\mathfrak{n}$ (here by support we mean set-theoretic rather than scheme-theoretic support, so the coherent sheaf $F(X)$ may be annihilated by some power of the ideal of $\mathfrak{n}$). Thus if we assume for a moment that the functor F can be lifted to a triangulated functor $\tilde{F}' : D^b(u_q - mod) \to D^b(Coh(\mathcal{N}))$, then (4) and Theorem 1 would yield a morphism from the left-hand side to the right-hand side of (14). Here we say that $\tilde{F}'$ is a lifting of F if $F \cong R\Gamma \circ \tilde{F}'$, where $R\Gamma(\mathcal{F}) = \bigoplus_i H^i(\mathcal{F})$ for $\mathcal{F} \in D^b(Coh(N))$.

It is easy to see that such a functor $\tilde{F}'$ does not exist. A meaningful version of the argument is as follows. Let $\mathbf{O}$ be the differential graded algebra $RHom_{u_q}(\mathbb{I}, \mathbb{I})$ (thus $\mathbf{O}$ is a well-defined object of the categroy of differential graded algebras with inverted quasiisomorphisms); the Ginzburg-Kumar theorem (13) shows that the cohomology algebra $H^\bullet(\mathbf{O}) \cong \mathcal{O}(\mathcal{N})$. Let $DGmod(\mathbf{O})$ be the triangulated category of differential graded modules over $\mathbf{O}$ with inverted quasiisomorphisms. Let $D \subset DGmod(\mathbf{O})$ be the full subcategory of DG-modules whose cohomology is a finitely generated module over $H^\bullet(\mathbf{O}) = \mathcal{O}(\mathcal{N})$, and let $D_{\infty/2} \subset D$ be the full triangulated

subcategory of DG-modules, whose cohomology is a coherent sheaf on $\mathcal{N}$ supported (set-theoretically) on $\mathfrak{n}$.

We have a functor $\tilde{F} : D^b(u_q - mod) \to D$ given by $\tilde{F} : X \mapsto RHom(\mathbb{I}, X)$. It is easy to see that $\tilde{F}$ sends complexes of finite homological dimension over b_q to $D_{\infty/2}$; and that $\tilde{F}(\mathbb{I}) = \mathbf{O}$. Thus, by Theorem 1, (4) provides a morphism

$$Ext^{\infty/2+\bullet}(\mathbb{I}, \mathbb{I}) \longrightarrow Hom^{\bullet}_{D_{D_{\infty/2}}}(\mathbf{O}, \mathbf{O}).$$

One can then show that this morphism is an isomorphism; and also that the DG-algebra $\mathbf{O}$ is *formal* (quasi-isomorphic to the DG-algebra $H^\bullet(\mathbf{O})$ with trivial differential), which implies that

$$Hom^{\bullet}_{D_{D_{\infty/2}}}(\mathbf{O}, \mathbf{O}) \cong H^{\bullet}_{\mathfrak{n}}(\mathcal{N}, \mathcal{O})$$

(notice that the latter isomorphism is not compatible with homological gradings). This yields the isomorphism (14).

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